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Discrete Mathematics 146 (1995) 325–328

DISCRETE MATHEMATICS

Note

A not 3-choosable planar graph without 3-cycles

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Received 23 March 1994, revised 6 April 1994

Abstract

An *L-list coloring* of a graph G is a proper vertex coloring in which every vertex v receives a color from a prescribed list $L(v)$. G is called *k-choosable* if all lists $L(v)$ have the cardinality k and G is L-list colorable for all possible assignments of such lists.

Recently, Thomassen has proved that every planar graph with girth greater than 4 is 3-choosable. Furthermore, it is known that the chromatic number of a planar graph without 3-cycles is at most 3. Consequently, the question resulted whether every planar graph without 3-cycles is 3-choosable.

In the following we will give a planar graph without 3-cycles which is not 3-choosable.

1. Introduction

One of the most interesting variations of ordinary graph colorings is the L-list coloring of the vertices of graph [7].

In practical applications [6] it is often required to choose the color for a vertex v from a *list* $L(v)$ of allowed colors. A graph $G = G(V, E)$ is called *L-list colorable* if there is a coloring f of the vertices of G with

1. $f(u) \neq f(v)\ \forall (u, v) \in E(G)$,
2. $f(v) \in L(v)\ \forall v \in V(G)$.

G is called *k-choosable* if G is L-list colorable for every assignment of lists $L(v)$ where each $L(v)$ has exactly k elements.

The conception of L-list coloring, choosability and choice number (the smallest k so that G is k-choosable) was introduced by both Vizing [10], 1976 and Erdős, Rubin and Taylor [2], 1979.

During the last year, some new results were found about the choosability of planar graphs. Alon and Tarsi [1] discovered that every planar bipartite graph is 3-choosable. Two conjectures of Erdős et al. [2], namely 'Every planar graph is 5-choosable'

SSDI 0012-365X(94)00180-4

and 'there are planar graphs which are not 4-choosable' were proved by Thomassen [8] and Voigt [11].

The investigation of planar 3-choosable graphs is a very interesting topic in this connection. The class of 2-choosable graphs was completely characterized by Erdős et al. [2] but there are no ideas about a characterization of 3-choosable graphs. Mahadev et al. [5] considered 3-choosable complete bipartite graphs. However, not even these graphs have been characterized completely. Concerning planar graphs, Kratochvil and Tuza [4] raised the question whether every planar graph without 3-cycles is 3-choosable. Recently, Thomassen [9] proved that every planar graph with girth greater than 4 (without 3- and 4-cycles) is 3-choosable. On the other hand, we have the well-known result of Grötzsch [3] that every planar graph without 3-cycles is 3-colorable. To complete these results it was very interesting to investigate the choosability of planar graphs with girth equal to 4.

In the following, we will construct a planar graph with girth 4 which is not 3-choosable.

2. Construction and list assignment

Remark. In the following, the colors are denoted by $1, 2, 3 \ldots$ or $a, b, c, \ldots$

First, we consider a vertex P with $L(P) = (1, 2, 3)$ and add three times the graph of Fig. 1 with $a = 1$, $a = 2$ and $a = 3$.

Assume the vertex P is colored with color a. We obtain inevitably one of the following colorings:

1. 4-cycle $PEFH$: $a, 6, 4, 6$,
2. 4-cycle $PEFK$: $a, 6, 5, 6$,
3. 4-cycle $PECD$: $a, 7, 4, 7$,
4. 4-cycle $PECB$: $a, 7, 5, 7$.

Now, we insert new vertices and edges (Fig. 2) inside (respectively outside) for each of these 4-cycles. The list assignment for the new vertices depends on the possible

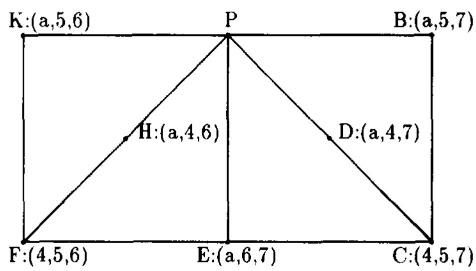

Fig. 1.

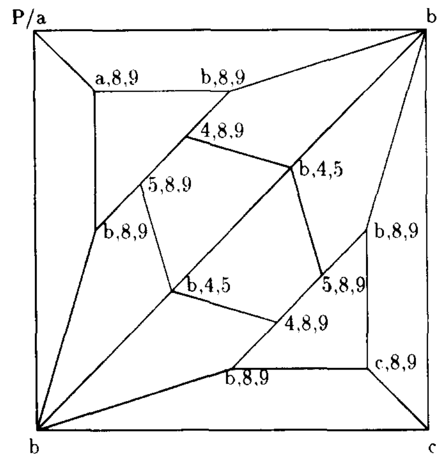

Fig. 2.

coloring of the current 4-cycle which is given in (1–4) and denoted by a, b, c, b in the graph shown in Fig. 2.

The resulting graph is denoted by G and has $22 + 12 * 12 = 166$ vertices.

3. Choosability of G

Theorem. *The graph G is planar, 3-cycle-free and not 3-choosable.*

Proof. Because of the construction it is easy to see that G is planar and 3-cycle-free.

We consider the list assignment given in Figs. 1 and 2 to prove that G is not 3-choosable. W.l.o.g let vertex P be colored with color 1. We consider the suitable Fig. 1 with $a = 1$. It follows inevitably: one of the following 4-cycles has the specified coloring:

1. 4-cycle $PEFH$: $1, 6, 4, 6$,
2. 4-cycle $PEFK$: $1, 6, 5, 6$,
3. 4-cycle $PECD$: $1, 7, 4, 7$,
4. 4-cycle $PECB$: $1, 7, 5, 7$.

W.l.o.g. let the 4-cycle $PEFH$ be colored with colors $1, 6, 4, 6$. Now we consider Fig. 2 for the 4-cycle $PEFH$ with $a = 1, b = 6, c = 4$. We ascertain that one of the 5-cycles of this figure is not properly colorable with colors of the given lists. □

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